PRESSURE OF A CIRCULAR DIE [PUNCH] ON AN ELASTIC HALF-SPACE, WHOSE MODULUS OF ELASTICITY IS AN EXPONENTIAL FUNCTION OF DEPTH

(DAVLENIE KRUGLOGO SHTAMPA NA UPRUGOE POLUPROSTRANSTVO, MODUL' UPRUGOSTI KOTOROGO IAVLIAETSIA STEPENNOI FUNKTSICI GLUBINY)

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In a paper by B.G. Korenev [1], one finds the formulation of the problem of a die, resting on a foundation whose modulus of elasticity varies with depth according to an exponential law. For an axisymmetric case, as indicated in reference [1], the pressure on the surface of the half-space and its settlement can be expressed, respectively, in the form

$$p_0(\rho) = \int_0^\infty f(\beta) J_0(\beta\rho)\, d\beta, \qquad W_0(\rho) = A_m \int_0^\infty \beta^\alpha f(\beta) J_0(\beta\rho)\, d\beta \tag{0.1}$$

where J_0 is a Bessel function, m is the exponential index in the expression for the elastic nucleus $(1 > m > 0)$,

$$A_m = \frac{2^{1-m}\Gamma^{1}/_{2}(1-m)}{E_m \Gamma^{1}/_{2}(1+m)}, \qquad \alpha = m - 1$$

For $m = 0$, one obtains the ordinary homogeneous half-space and in this case $E_0 = E/(1-\mu^2)$.

In the following, a more convenient solution of the problem is presented, which is better suited for practical calculations; one also will find corrections regarding certain inaccuracies in reference [1].

1. Assume the radius of the die equals unity, which always can be accomplished by introducing a dimensionless coordinate. Then the problem of determining the pressure under the die reduces to the problem of solving a "coupled" integral equation

$$\int_0^\infty \beta^\alpha f(\beta) J_0(\beta\rho)\, d\beta = g_0(\rho) \qquad \left(g_0(\rho) = \frac{W_0(\rho)}{A_m},\ 0 < \rho < 1\right) \tag{1.1}$$

$$\int_0^\infty f(\beta) J_0(\beta\rho)\, d\beta = 0 \qquad (1 < \rho < \infty)$$

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For a die whose surface after impression is determined by equation

$$z = w_n(\rho)\cos n\varphi \tag{1.2}$$

where ϕ is the polar angle, the pressure may be determined from the formula

$$p(\rho, \varphi) = p_n(\rho)\cos n\varphi \tag{1.3}$$

Thereby, equations (0.1) and (1.2) and the "coupled" integral equation (1.1) retain their validity, except that subscript 0 has to be replaced throughout by subscript n.

2. We have the following "coupled" integral equation

$$\int_0^\infty \beta^\alpha f(\beta) \int_{J_n} (\beta\rho)\, d\beta = g_n(\rho) \quad (0<\rho<1), \qquad \int_0^\infty f(\beta) \int_{J_n} (\beta\rho)\, d\beta = 0 \quad (1<\rho<\infty) \tag{2.1}$$

We may use the following relations, well known in the theory of Bessel functions

$$\int_{J_n} (\beta\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{-s}\Gamma(1/2 - 1/2 s + 1/2 n)}{\Gamma(1/2 + 1/2 s + 1/2 n)} \rho^{s-1}\beta^{s-1} ds \tag{2.2}$$

$$\beta^\alpha \int_{J_n} (\beta\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{-s+\alpha}\Gamma(1/2 - 1/2 s + 1/2\alpha + 1/2 n)}{\Gamma(1/2 + 1/2 s + 1/2\alpha + 1/2 n)} \beta^{s-1}\rho^{s-\alpha-1}\, ds$$

We introduce the notations

$$\int_0^\infty f(\beta)\,\beta^{s-1} d\beta = F(s)$$

Then from (2.1) we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{2^{\alpha-s}\Gamma(1/2 - 1/2 s + 1/2\alpha + 1/2 n)}{\Gamma(1/2 + 1/2 s - 1/2\alpha + 1/2 n)} \rho^{s-\alpha-1} ds = g(\rho) \qquad (0<\rho<1)$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{2^{-s}\Gamma(1/2 - 1/2 s + 1/2 n)}{\Gamma(1/2 + 1/2 s + 1/2 n)} \rho^{s-1} ds = 0 \qquad (1<\rho<\infty) \tag{2.3}$$

Using the formulas

$$\int_0^z \rho^{2\gamma-1} (z^2 - \rho^2)^{\delta-1}\, dp = \frac{1}{2} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} z^{2\gamma+2\delta-2}$$

$$\int_0^z \bar{\rho}^{-2\gamma-2\delta+1} (\rho^2 - z^2)^{\delta-1}\, dp = \frac{1}{2} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} z^{-2\gamma} \tag{2.4}$$

we obtain from (2.3)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{2^{\alpha-s}\Gamma(1/2 - 1/2 s + 1/2\alpha + 1/2 n)\,\Gamma(1/2\alpha + 1)}{(\Gamma(1/2 + 1/2 s + 1/2 n))} x^s ds = An(x) \tag{2.5}$$

where

$$An(x) = x^{-n} \frac{d}{dx} \int_0^x g_n(\rho) \rho^{n+1} (x^2 - \rho^2)^{1/2\alpha} d\rho \qquad (0 < x < 1) \tag{2.6}$$
$$An(x) = 0 \qquad (1 < x < \infty)$$

Applying second formula (2.4) to (2.5) we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{2^{\alpha-s-1} \Gamma(1/2 + 1/2n - 1/2s) \Gamma(1/2\alpha + 1)^2}{\Gamma(1/2 + 1/2s + 1/2n)} r^{s-n+1} ds =$$
$$= \int_r^\infty An(x)\, x^{-\alpha-n} (x^2 - r^2)^{\frac{\alpha}{2}} dx$$

Making use of the formula

$$\Gamma\left(\frac{1}{2} + \frac{1}{2}n - \frac{1}{2}s\right) = \frac{1}{2}(-1 + n - s)\Gamma\left(-\frac{1}{2} + \frac{1}{2}n - \frac{1}{2}s\right) \tag{2.7}$$

we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{2^{\alpha-s} \Gamma(1/2 + 1/2n - 1/2s) \Gamma(1/2\alpha + 1)^2}{\Gamma(1/2 + 1/2s + 1/2n)} r^{s-n} ds =$$
$$= -\frac{d}{dr} \int_r^\infty A^n(x)\, x^{-\alpha-n} (x^2 - r^2)^{1/2\alpha} dx \tag{2.8}$$

The final expression for $P_n(r)$ has the form

$$P_n(r) = -\frac{2^{-\alpha}}{\Gamma(1/2\alpha + 1)^2} r^{n-1} \frac{d}{dr} \int_r^a \frac{x^{-\alpha-2n} dx}{(x^2 - r^2)^{-1/2\alpha}} \frac{d}{dx} \int_0^x \frac{g_n(\rho) \rho^{n+1} d\rho}{(x^2 - \rho^2)^{-1/2\alpha}} \tag{2.9}$$

3. By way of an example we consider the pressure of a plane circular die on an elastic half-space. Assume a force P acting on the die along its axis, in which case its deformation is constant and equal W_0. The formula (2.9) then reads

$$P_0(r) = -\frac{g_0 2^{-\alpha}}{\Gamma(1/2\alpha + 1)^2} \frac{1}{r} \frac{d}{dr} \int_r^a \frac{(x^2 - r^2)^{1/2\alpha} dx}{x^\alpha} \frac{d}{dx} \int_0^x \rho (x^2 - \rho^2)^{1/2\alpha} d\rho$$

On the basis of (2.4)

$$\int_0^x \rho (x^2 - \rho^2)^{1/2\alpha} d\rho = \frac{1}{\alpha + 2} x^{\alpha+2}$$

Then

$$P_0(r) = -\frac{g_0 2^{-\alpha}}{\Gamma(1/2\alpha + 1)^2} \frac{1}{r} \frac{d}{dr} \int_r^a x (x^2 - r^2)^{\frac{\alpha}{2}} dx$$

Whence

$$P_0(r) = \frac{g_0 2^{-\alpha}}{\Gamma(1/2\alpha + 1)^2} (a^2 - r^2)^{1/2\alpha}$$

Or finally

$$P_0(r) = \frac{W_0 E_m}{\Gamma(^1/_2(1-m))\Gamma(^1/_2(1+m))}(a^2 - r^2)^{^1/_2(m-1)}$$

The formula for the pressure under a plane circular die proposed by B.G. Korenev [1] is incorrect.

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